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ISOSPECTRAL CLOSED RIEMANNIAN MANIFOLDS WHICH ARE NOT LOCALLY ISOMETRIC

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Two compact Riemannian manifolds are said to be *isospectral* if the associated Laplace-Beltrami operators have the same eigenvalue spectrum. Milnor [18] constructed the first pair of isospectral, nonisometric manifolds, a pair of 16-dimensional flat tori. Many new examples and also techniques for constructing examples have appeared in the past decade; see for example [2], [3], [4], [6], [10], [11], [12], [13], [22] and [25] or the surveys [1], [2], [5] and [8]. However, in all these examples, the isospectral manifolds are locally isometric; in particular, in all the examples of isospectral closed Riemannian manifolds, the manifolds have a common Riemannian covering.

Recently, Zoltan Szabo [24] constructed the first examples of isospectral Riemannian manifolds (with boundary) which are not locally isometric. The manifolds are geodesic balls in different harmonic manifolds of nonpositive curvature. These manifolds are first introduced in [23].

The purpose of this article is to construct pairs of isospectral *closed* Riemannian manifolds with no common covering. The manifolds involved are two-step nilmanifolds of Heisenberg type. In each case, the construction gives two continuous families F_1 and F_2 of Riemannian manifolds all of which are isospectral. Those in a given family are locally isometric but not isometric. The manifolds in F_1 are not locally isometric to those in F_2. (See Remark 2.2.)

After describing the general construction, we will give specific examples and describe the geometry of two of the pairs of isospectral but not locally isometric manifolds. We will note differences both in the size of the isometry groups of the simply connected covers and in the curvature.

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1. Lie algebras of Heisenberg type

A Riemannian nilmanifold $(\Gamma\backslash G, g)$ is a quotient of a simply connected nilpotent Lie group G by a discrete (possibly trivial) subgroup Γ, together with a Riemannian metric g whose lift to G is left-invariant. We will abuse notation and call the metric on $\Gamma\backslash G$ *left-invariant*. We say the nilmanifold is of *step size* k if G is k-step nilpotent.

The metric g is uniquely associated with an inner product $\langle\ ,\ \rangle$ on the Lie algebra $\mathfrak{g}$ of G. We will call $(\mathfrak{g}, \langle\ ,\ \rangle)$ a *metric Lie algebra.* Two metric Lie algebras $(\mathfrak{g}_1, \langle\ ,\ \rangle_1)$ and $(\mathfrak{g}_2, \langle\ ,\ \rangle_2)$ will be said to be *isomorphic* if there exists a Lie algebra isomorphism from $\mathfrak{g}_1$ to $\mathfrak{g}_2$ which is also an inner product space isometry.

Proposition 1.1 (Wilson [26]). *Two simply connected Riemannian nilmanifolds are isometric if and only if the associated metric Lie algebras are isomorphic.*

Notation 1.2. Consider a two-step nilpotent metric Lie algebra $(\mathfrak{g}, \langle\ ,\ \rangle)$. Since $\mathfrak{g}$ is two-step nilpotent, the derived algebra $\mathfrak{z} = [\mathfrak{g}, \mathfrak{g}]$ lies in the center of $\mathfrak{g}$. Let $\mathfrak{v}$ denote the orthogonal complement of $\mathfrak{z}$. Define a map $j: \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ by

$$\langle j(Z)(X), Y\rangle = \langle [X, Y], Z\rangle \tag{1.3}$$

for all $Z \in \mathfrak{z}$ and $X, Y \in \mathfrak{v}$. Following Kaplan [14], we say the two-step nilpotent metric Lie algebra $(\mathfrak{g}, \langle\ ,\ \rangle)$ is of *Heisenberg type* if

$$j(Z)^2 = -|Z|^2 \operatorname{Id}. \tag{1.4}$$

This condition implies that the derived algebra $\mathfrak{z}$ of $\mathfrak{g}$ coincides with the center of $\mathfrak{g}$. We say the nilmanifold $(\Gamma\backslash G, g)$ is of *Heisenberg type* if the associated metric Lie algebra is of Heisenberg type.

The geometry of simply connected nilmanifolds of Heisenberg type was studied in [7], [15], and [17].

The metric Lie algebras of Heisenberg type have been completely classified [21]. We now describe this classification. Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be of Heisenberg type and let $j: \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ be the associated representation of the center $\mathfrak{z}$ of $\mathfrak{g}$. Let $C(\mathfrak{z})$ denote the Clifford algebra associated to the bilinear form $-\langle\ ,\ \rangle$ on $\mathfrak{z}$. Using (1.4), one sees that j induces an action of $C(\mathfrak{z})$ on $\mathfrak{v}$.

Conversely, given an inner product space $(\mathfrak{z}, \langle\ ,\ \rangle)$ and a representation $T: C(\mathfrak{z}) \to \operatorname{End}(\mathfrak{v})$ of $C(\mathfrak{z})$ on a finite-dimensional vector space $\mathfrak{v}$, define $j: \mathfrak{z} \to \operatorname{End}(\mathfrak{v})$ by restriction of T. Then (1.4) is automatically satisfied, and we can define an inner product on $\mathfrak{v}$ so that the unit sphere in $\mathfrak{z}$ acts orthogonally. Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be the orthogonal direct sum of the inner product spaces $\mathfrak{z}$ and $\mathfrak{v}$, and define $[\ ,\ ]\colon \mathfrak{v}\times\mathfrak{v}\to\mathfrak{z}$ by formula (1.3). Extend $[\ ,\ ]$ to a Lie bracket on $\mathfrak{g}$ by requiring that $\mathfrak{z}$ lie in the center of $\mathfrak{g}$. Then under this structure, $(\mathfrak{g}, \langle\ ,\ \rangle)$ is a two-step nilpotent metric Lie algebra of Heisenberg type.

Every representation of a Clifford algebra is completely reducible. The inner product $\langle\ ,\ \rangle$ is uniquely determined up to a scalar on each irreducible submodule of $\mathfrak{v}$ by the condition that the unit sphere in $\mathfrak{z}$ acts orthogonally. Moreover, one checks easily that the isomorphism class of the metric Lie algebra is independent of the choice of such scalars. Thus every finite-dimensional representation of $C(\mathfrak{z})$ determines a unique (up to isomorphism) metric Lie algebra of Heisenberg type, and thus by Proposition 1.1, a unique simply connected nilmanifold of Heisenberg type. We caution though that the metric Lie algebras of Heisenberg type determined by inequivalent representations of $C(\mathfrak{z})$ may still be isomorphic.

Proposition 1.5 (see [21]). *If* $\dim(\mathfrak{z})$ *is not congruent to* 3 mod 4, *then* $C(\mathfrak{z})$ *has a unique (up to equivalence) irreducible representation. If* $\dim(\mathfrak{z})$ *is congruent to* 3 mod 4, *then* $C(\mathfrak{z})$ *has exactly two inequivalent irreducible representations, both of the same dimension.*

In the sequel we will be concerned only with the case in which $\dim(\mathfrak{z})$ is congruent to 3 mod 4. We now describe all examples in which $\dim(\mathfrak{z})$ is 3 or 7.

Example 1.6. Let A denote either the quaternion algebra or the Cayley algebra. Let $\mathfrak{z}$ be the pure quaternions (respectively, the pure Cayley numbers) with the standard inner product, and let $\mathfrak{v}$ be the orthogonal direct sum of k copies of A with the standard inner product. Choose nonnegative integers a and b with $k = a + b$. Define the map $j\colon \mathfrak{z}\to \mathfrak{so}(\mathfrak{v})$ by

$$j(Z)(X_1, \cdots, X_a, Y_1, \cdots, Y_b) = (ZX_1, \cdots, ZX_a, Y_1Z, \cdots, Y_bZ) \tag{1.7}$$

where ZX_i and Y_jZ denote multiplication in A. Then j satisfies equation (1.4), and thus the resulting two-step nilpotent metric Lie algebra $(\mathfrak{g}_{a,b}, \langle\ ,\ \rangle_{a,b})$ is of Heisenberg type. (We will use the same notation $\mathfrak{g}_{a,b}$ for the Lie algebras constructed from either the quaternions or the Cayley numbers. Whenever we discuss pairs of such algebras below, we assume implicitly that they are both constructed using the same algebra A.) The metric Lie algebras $(\mathfrak{g}_{a,b}, \langle\ ,\ \rangle_{a,b})$ and $(\mathfrak{g}_{a',b'}, \langle\ ,\ \rangle_{a',b'})$ are isomorphic if and only if (a, b) coincides with (a', b') up to order. (The isomorphism between $\mathfrak{g}_{1,0}$ and $\mathfrak{g}_{0,1}$ is given by $X+Z \to \overline{X}-Z$ for $X \in \mathfrak{v}$ and $Z \in \mathfrak{z}$.)

2. Isospectral manifolds of Heisenberg type

Let $M = (\Gamma\backslash G, g)$ be a compact two-step Riemannian nilmanifold. The exponential map from $\mathfrak{g}$ to G is a diffeomorphism; denote its inverse by log. While $\log(\Gamma)$ need not be a lattice in $\mathfrak{g}$, it intersects the derived algebra $\mathfrak{z}$ of $\mathfrak{g}$ in a lattice of maximal rank $L_{\mathfrak{z}}$ and it projects to a lattice L' of maximal rank in $\mathfrak{g}/\mathfrak{z}$. We can identify $\mathfrak{g}/\mathfrak{z}$ with the orthogonal complement $\mathfrak{v}$ of $\mathfrak{z}$ in $\mathfrak{g}$ and L' with a lattice $L_{\mathfrak{v}}$ in $\mathfrak{v}$.

The compact two-step nilmanifold $M = \Gamma\backslash G$ is a principal torus bundle over a torus. The fibration arises from the fibration $Z(G) \to G \to G/Z(G)$ of G, where $Z(G)$ is the (necessarily central) derived group of G. We will denote the fiber and base tori by T_F and T_B, respectively. The left-invariant Riemannian metric g on G induces flat metrics g_F and g_B on T_F and T_B so that $\Gamma\backslash G \to T_B$ is a Riemannian submersion with totally geodesic fibers. The tori T_F and T_B are isometric to the tori $L_{\mathfrak{z}}\backslash\mathfrak{z}$ and $L_{\mathfrak{v}}\backslash\mathfrak{v}$ with the metrics arising from the inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$.

Theorem 2.1. *Let $M = (\Gamma\backslash G, g)$ and $M' = (\Gamma'\backslash G', g')$ be two-step Riemannian nilmanifolds of Heisenberg type. Suppose that the corresponding pairs of flat tori are isospectral; i.e.,* $\mathrm{spec}(T_F) = \mathrm{spec}(T_{F'})$ *and* $\mathrm{spec}(T_B) = \mathrm{spec}(T_{B'})$. *Then M is isospectral to M'.*

Remarks 2.2. (1) Let (M, g) be a Riemannian nilmanifold of Heisenberg type, and let k and m be the dimensions of the associated tori T_B and T_F. The construction in [11] shows that the metric g lies in a continuous $k(m-1)$ parameter family of isospectral, nonisometric (but locally isometric) Riemannian metrics on M. Thus Theorem 2.1 gives a pair of continuous families of metrics, all isospectral, such that those in the first family are not locally isometric to those in the second.

(2) The special case of the theorem in which the centers of G and G' are one-dimensional, i.e., G and G' are $(2n+1)$-dimensional Heisenberg groups, was proven in [12]. In this case M and M' have a common Riemannian cover, i.e., (G, g) is isometric to (G', g').

(3) In all the examples below, the pairs of flat tori will actually be isometric.

(4) Eberlein [7] proved that the condition $\mathrm{spec}(T_F) = \mathrm{spec}(T_{F'})$ and $\mathrm{spec}(T_B) = \mathrm{spec}(T_{B'})$ is necessary and sufficient for the pair of two-step

nilmanifolds M and M' of Heisenberg type to have the same length spectrum. (However, if one counts multiplicities in the length spectrum by the number of free homotopy classes of closed curves containing a geodesic of the given length rather than the total number, necessarily infinite, of geodesics of that length, then this condition is not sufficient. See [9] for details.)

The proof is by an explicit calculation of the spectra. Using the Kirillov theory of representations of a nilpotent Lie group, Pesce [19] computed the eigenvalues of an arbitrary compact two-step nilmanifold. We first summarize his results.

Let $M = (\Gamma\backslash G, g)$ be a compact two-step Riemannian nilmanifold. Recall that the Laplacian of (M, g) is given by $\Delta = -\sum_i X_i^2$ where $X_1, X_2, \cdots, X_n$ is an orthonormal basis of $\mathfrak{g}$ relative to the inner product $\langle\ ,\ \rangle$ defined by g. Letting $\rho = \rho_\Gamma$ denote the right action of G on $L^2(M)$, then the Laplacian acts on $L^2(M)$ as $\Delta = -\sum_i \rho_* X_i^2$.

Given any unitary representation (V, π) of G, we may define a Laplace operator $\Delta_{g,\pi}$ on V by $\Delta_{g,\pi} = -\sum_i \pi_* X_i^2$. The eigenvalues of this operator depend only on g and the equivalence class of the representation π. The space $(L^2(M), \rho)$ is the countable direct sum of irreducible representations (V_α, π_α), each occurring with finite multiplicity. The spectrum of M is the disjoint union of the spectra of the operators Δ_{g,π_α}.

Kirillov [16] showed that the equivalence classes of irreducible unitary representations of the simply connected nilpotent Lie group G are in one-to-one correspondence with the orbits of the coadjoint action of G on the dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G. We will denote the representation corresponding to the coadjoint orbit of $\lambda \in \mathfrak{g}^*$ by π_λ.

Richardson [20] computed the decomposition of $L^2(\Gamma\backslash G)$ into irreducible representations π_λ for an arbitrary compact nilmanifold. In case G is two-step nilpotent, Pesce obtained the following more explicit description.

Notation 2.3. Given $\lambda \in \mathfrak{g}^*$, define $B_\lambda : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ by

$$B_\lambda(X, Y) = \lambda([X, Y]).$$

Let $\mathfrak{g}_\lambda = \ker(B_\lambda)$ and let $\widetilde{B}_\lambda$ be the nondegenerate skew-symmetric bilinear form induced by B_λ on $\mathfrak{g}/\mathfrak{g}_\lambda$.

The image of $\log(\Gamma)$ in $\mathfrak{g}/\mathfrak{g}_\lambda$ is a lattice, which we denote by L_λ.

We will write $\Delta_{g,\lambda}$ for Δ_{g,π_λ}.

Proposition 2.4 [19]. *Let $\Gamma\backslash G$ be a compact two-step nilmanifold, let $\mathfrak{g}$ be the Lie algebra of G, and let $\lambda \in \mathfrak{g}^*$. Then π_λ appears in the* *quasiregular representation* ρ_Γ *of* G *on* $L^2(\Gamma\backslash G)$ *if and only if* $\lambda(\log(\Gamma) \cap \mathfrak{g}_\lambda) \subseteq \mathbb{Z}$. *In this case the multiplicity of* π_λ *is* $m_\lambda = 1$ *if* $\lambda(\mathfrak{z}) = \{0\}$, *and* $m_\lambda = (\det \widetilde{B}_\lambda)^{1/2}$ *otherwise, where the determinant is computed with respect to a lattice basis of* L_λ.

Let $\langle\ ,\ \rangle$ be the inner product on $\mathfrak{g}^*$ defined by the Riemannian inner product on $\mathfrak{g}$.

Proposition 2.5 [19]. (a) *If* $\lambda(\mathfrak{z}) = 0$, *then* π_λ *is a character of* G *and*

$$\operatorname{spec}(\Delta_{g,\lambda}) = \{4\pi^2\|\lambda\|^2\}.$$

(b) *If* $\lambda(\mathfrak{z}) \neq \{0\}$, *let* $\pm(-1)^{1/2}d_1, \cdots, \pm(-1)^{1/2}d_r$ *be the eigenvalues of* $\widetilde{B}_\lambda$. *Then*

$$\operatorname{spec}(\Delta_{g,\lambda}) = \{\mu(\lambda, p, g) : p \in \mathbb{N}^r\}$$

where

$$\mu(\lambda, p, g) = 4\pi^2 \sum_{i=1,\cdots,l} \lambda(Z_i)^2 + 2\pi \sum_{j=1,\cdots,r} (2p_j + 1)d_j$$

with $\{Z_1, \cdots, Z_l\}$ *a* g*-orthonormal basis of* $\mathfrak{g}_\lambda$. *The multiplicity of an eigenvalue* μ *is the number of* $p \in \mathbb{N}^r$ *such that* $\mu = \mu(\lambda, p, g)$.

Proof of Theorem 2.1. Let $M = (\Gamma\backslash G, g)$ be a two-step nilmanifold of Heisenberg type. Notice that the derived algebra $\mathfrak{z}$ of $\mathfrak{g}$ coincides with the center of $\mathfrak{g}$. Let

$$A_1(\Gamma) = \{\lambda \in \mathfrak{g}^* : \lambda(\mathfrak{z}) = 0 \text{ and } \lambda(\log(\Gamma)) \subset \mathbb{Z}\}$$

and

$$A_2(\Gamma) = \{\lambda \in \mathfrak{g}^* : \lambda(\mathfrak{z}) \neq 0 \text{ and } \lambda(\log(\Gamma) \cap \mathfrak{g}_\lambda) \subset \mathbb{Z}\}.$$

By Proposition 2.4, $\operatorname{spec}(M)$ is the union, with appropriate multiplicities, of the $\operatorname{spec}(\Delta_{g,\lambda})$ as λ varies over $A_1(\Gamma) \cup A_2(\Gamma)$. Let S_1 and S_2 be the parts of the spectrum corresponding to $A_1(\Gamma)$ and $A_2(\Gamma)$, respectively. By Proposition 2.5(a), S_1 is precisely the spectrum of the associated flat torus T_B.

Now consider S_2. For $\lambda \in A_2(\Gamma)$, we have $\mathfrak{z} = \mathfrak{g}_\lambda$ and $L_\mathfrak{z} = \log(\Gamma) \cap \mathfrak{g}_\lambda$. The coadjoint orbit of λ consists of all elements of $\mathfrak{g}^*$ which agree with λ on $\mathfrak{z}$. Thus the map τ: $A_2(\Gamma) \to \mathfrak{z}^*$ given by $\tau(\lambda) = \lambda_{|_\mathfrak{z}}$ induces a bijection between the set of coadjoint orbits in $A_2(\Gamma)$ and the dual lattice $L_\mathfrak{z}^*$ of $L_\mathfrak{z}$ in $\mathfrak{z}^*$. (Recall that $L_\mathfrak{z}^* = \{\alpha \in \mathfrak{z}^* : \alpha(X) \in \mathbb{Z} \text{ for all } X \in L_\mathfrak{z}\}$.) Given $\lambda \in A_2(\Gamma)$, let $Z_\lambda \in \mathfrak{z}$ satisfy $\lambda(Z) = \langle Z_\lambda, Z\rangle$ for all $Z \in \mathfrak{z}$. For

X, Y in the orthogonal complement $\mathfrak{v}$ of $\mathfrak{z}$, we have

$$B_\lambda(X, Y) = \langle [X, Y], Z_\lambda \rangle = \langle j(Z_\lambda)X, Y \rangle.$$

Since $\mathfrak{v} \cong \mathfrak{g}/\mathfrak{g}_\lambda$, the eigenvalues of $\widetilde{B}_\lambda$ are the eigenvalues of $j(Z_\lambda)$, namely $\pm(-1)^{1/2}\|Z_\lambda\|$. Thus in the notation of Proposition 2.5, $r = (1/2)\dim(\mathfrak{v})$ and $d_i = \|Z_\lambda\|$ for $i = 1, \cdots, r$. Moreover, since $\mathfrak{z} = \mathfrak{g}_\lambda$, we have

$$\sum_{i=1,\cdots,l} \lambda(Z_i)^2 = \sum_{i=1,\cdots,l} \langle Z_i, Z_\lambda \rangle^2 = \|Z_\lambda\|^2.$$

Thus

$$\begin{aligned}\mu(\lambda, p, g) &= 4\pi^2\|Z_\lambda\|^2 + 2\pi \sum_{j=1,\cdots,r} (2p_j + 1)\|Z_\lambda\| \\ &= 4\pi^2\|\tau(\lambda)\|^2 + 2\pi \sum_{j=1,\cdots,r} (2p_j + 1)\|\tau(\lambda)\|\end{aligned}$$

where $\|\tau(\lambda)\|$ is computed with respect to the inner product on $\mathfrak{z}^*$ induced by the Riemannian inner product on $\mathfrak{z}$. Thus $\operatorname{spec}(\Delta_{g,\lambda})$ depends only on $\|\tau(\lambda)\|$.

We use Proposition 2.4 to compute the multiplicity m_λ of π_λ in $L^2(\Gamma\backslash G)$. With respect to an orthonormal basis $\mathscr{B}$ of $\mathfrak{v}$, we have $\det[\widetilde{B}_\lambda]_{\mathscr{B}} = \|Z_\lambda\|^{2r} = \|\tau(\lambda)\|^{2r}$. Consequently, m_λ, the square root of the determinant with respect to a lattice basis of L_λ (viewed as a lattice in $\mathfrak{v}$), is given by

$$m_\lambda = \|\tau(\lambda)\|^r(\operatorname{Vol}(T_B))$$

where the volume is with respect to the metric g_B on T_B.

Now $\operatorname{spec}(T_F)$ is the collection of numbers $4\pi^2\|\alpha\|^2$ as α ranges over $L_{\mathfrak{z}}^*$, counted with multiplicities. Equivalently, $\operatorname{spec}(T_F)$ is the collection of numbers $4\pi^2\|\tau(\lambda)\|^2$ as λ ranges over $A_2(\Gamma)$. The multiplicity of an element γ of $\operatorname{spec}(T_F)$ is the number of coadjoint orbits of elements λ in $A_2(\Gamma)$ for which $\gamma = 4\pi^2\|\tau(\lambda)\|^2$. Since, moreover, $\operatorname{Vol}(T_B)$ depends only on $\operatorname{spec}(T_B)$, we see that both the eigenvalues in S_2 and their multiplicities are determined by $\operatorname{spec}(T_B)$ and $\operatorname{spec}(T_F)$. The theorem follows.

Examples 2.6. Fix a choice A of quaternion algebra or Cayley algebra and let $(\mathfrak{g}_{a,b}, \langle\ ,\ \rangle_{a,b})$ be the Lie algebra of Heisenberg type constructed in Example 1.6. Let $G_{a,b}$ be the corresponding simply connected Lie group and $g_{a,b}$ the Riemannian metric on $G_{a,b}$ associated with $\langle\ ,\ \rangle_{a,b}$.

One way to construct a cocompact discrete subgroup Γ of $G_{a,b}$ is to let $L_{a,b}$ be the lattice in $\mathfrak{g}_{a,b}$ spanned by the standard basis elements.

Then the image of $L_{a,b}$ under the Lie group exponential map generates a cocompact discrete subgroup $\Gamma_{a,b}$ of $G_{a,b}$. Theorem 2.1 implies:

Theorem 2.7. *If $a+b = a'+b'$, then the nilmanifolds $M_{a,b} = (\Gamma_{a,b}\backslash G_{a,b}, g_{a,b})$ and $M_{a',b'} = (\Gamma_{a',b'}\backslash G_{a',b'}, g_{a',b'})$ are isospectral. They are locally isometric if and only if (a', b') coincides with (a, b) up to order.*

Remark. The simply connected nilmanifolds $(G_{a,b}, g_{a,b})$ constructed from the quaternions play a role in Szabo's construction referred to in the introduction. It is not clear whether there is any connection between the isospectrality of the nilmanifolds and the isospectrality of Szabo's examples.

3. Geometry of the examples

We will compare both the local and the global geometry of the simply connected covers $(G_{2,0}, g_{2,0})$ and $(G_{1,1}, g_{1,1})$ of the manifolds $M_{2,0}$ and $M_{1,1}$ of Example 2.6 and Theorem 2.7 for both the quaternion and the Cayley constructions.

Proposition 3.1. *The isometry group of $(G_{2,0}, g_{2,0})$ has higher dimension than that of $(G_{1,1}, g_{1,1})$.*

Proof. As shown in [26], the isometry group of any simply connected Riemannian nilmanifold (G, g) is the semidirect product of the group G of left translations with the group of all orthogonal automorphisms of (G, g). (An *orthogonal automorphism* of (G, g) is an automorphism ϕ of G satisfying $\Phi^* g = g$.) The Lie algebra of the full isometry group is then the semidirect sum $\mathfrak{g} + D(\mathfrak{g})$, where $D(\mathfrak{g})$ is the space of all skew-symmetric derivations of $(\mathfrak{g}, \langle\ ,\ \rangle)$. In the case of a nilmanifold of Heisenberg type,

$$D(\mathfrak{g}) \cong \mathfrak{so}(\mathfrak{z}) + \mathfrak{so}_{\mathfrak{z}}(\mathfrak{v})$$

where $\mathfrak{z}$ denotes the center of $\mathfrak{g}$, $\mathfrak{v}$ denotes the orthogonal complement of $\mathfrak{z}$ and

$$\mathfrak{so}_{\mathfrak{z}}(\mathfrak{v}) = \{\tau \in \mathfrak{so}(\mathfrak{v}) : \tau \circ j(Z) = j(Z) \circ \tau \text{ for all } Z \in \mathfrak{z}\}$$

(see [21, Theorem 6]). Thus we need only compare the dimensions of the spaces $\mathfrak{so}_{\mathfrak{z}}(\mathfrak{v})$ associated with $G_{2,0}$ and $G_{1,1}$.

If we view $\mathfrak{v}$ as a module over the Clifford algebra $C(\mathfrak{z})$, then

$$\mathfrak{so}_{\mathfrak{z}}(\mathfrak{v}) = \mathfrak{so}(\mathfrak{v}) \cap \operatorname{End}_{C(\mathfrak{z})}(\mathfrak{v})$$

where $\operatorname{End}_{C(\mathfrak{z})}(\mathfrak{v})$ is the space of Clifford module endomorphisms of $\mathfrak{v}$. In the case of $G_{1,1}$, the Clifford module $\mathfrak{v}$ is the direct sum of two inequivalent Clifford modules $\mathfrak{v}_1$ and $\mathfrak{v}_2$. Any $\tau \in \mathrm{End}_{C(\mathfrak{z})}(\mathfrak{v})$ must leave each of $\mathfrak{v}_1$ and $\mathfrak{v}_2$ invariant; i.e.,

$$\mathfrak{so}_{\mathfrak{z}}(\mathfrak{v}) = \mathfrak{so}_{\mathfrak{z}}(\mathfrak{v}_1) + \mathfrak{so}_{\mathfrak{z}}(\mathfrak{v}_2).$$

Thus, for example, if A is the quaternions, then

$$\mathfrak{so}_{\mathfrak{z}}(\mathfrak{v}) \cong \mathfrak{sp}(1) \times \mathfrak{sp}(1),$$

a six-dimensional Lie algebra. In this case $D(\mathfrak{g})$ is 9-dimensional and, since $\mathfrak{g}$ is 11-dimensional, the full isometry group has dimension 20. On the other hand, in the case of $G_{2,0}$, the Clifford module $\mathfrak{v}$ is the direct sum of the two equivalent Clifford modules and, if A is the quaternions, then $\mathfrak{so}_{\mathfrak{z}}(\mathfrak{v}) \cong \mathfrak{sp}(2)$, a 10-dimensional Lie algebra. $D(\mathfrak{g})$ is then 13-dimensional and the full isometry group has dimension 24. The Cayley case is similar.

Proposition 3.2 [21]. *Let A be the Cayley algebra. Then every geodesic in $(G_{2,0}, g_{2,0})$ is an orbit of a one-parameter group of isometries, whereas some geodesics in $(G_{1,1}, g_{1,1})$ are not orbits.*

In the case that A is the quaternions, all geodesics in both manifolds are orbits of one-parameter groups of isometries.

We now consider the curvature of these manifolds. We will say a subspace S of the tangent space at a point of a manifold is *flat* if every two-plane in S has sectional curvature zero. Recall that for any nilmanifold (G, g), the tangent space at an arbitrary point is identified with the Lie algebra $\mathfrak{g}$. The following result is elementary; see [7] for example.

Lemma 3.3. *Let (G, g) be a two-step Riemannian nilmanifold. Letting $\mathfrak{z}$ denote the center of the Lie algebra $\mathfrak{g}$ and $\mathfrak{v}$ the orthogonal complement of $\mathfrak{z}$, then $\mathfrak{v}$ (respectively, $\mathfrak{z}$) is the direct sum of all negative (respectively, nonnegative) eigenspaces of the Ricci tensor. The subspace $\mathfrak{z}$ is flat. In the case of Riemannian nilmanifolds of Heisenberg type, the Ricci tensor has only two eigenvalues, one negative and one positive; the eigenspaces are $\mathfrak{v}$ and $\mathfrak{z}$.*

Thus $\mathfrak{z}$ and $\mathfrak{v}$ are geometrically distinguished subspaces of $\mathfrak{g}$.

Proposition 3.4. *Let A be the quaternions. In the case of $(G_{1,1}, g_{1,1})$, the subspace $\mathfrak{v}$ of $\mathfrak{g}_{1,1}$ decomposes into an orthogonal direct sum $\mathfrak{v} = \mathfrak{v}_1 + \mathfrak{v}_2$ of two flat four-dimensional subspaces. On the other hand, in the case of $(G_{2,0}, g_{2,0})$, the subspace $\mathfrak{v}$ of $\mathfrak{g}_{2,0}$ contains no flat four-dimensional subspaces.*

Proof. In both cases, the sectional curvature of the 2-plane spanned by a pair of orthonormal vectors X and Y in $\mathfrak{v}$ is given by $K(X, Y) = (-3/4)\|[X, Y]\|^2$. (See [7].) Thus a subspace of $\mathfrak{v}$ is flat if and only if it is an abelian subalgebra of $\mathfrak{g}_{a,b}$. Moreover, $\mathfrak{v} = A \times A$ and $\mathfrak{z}$ is the space of pure quaternions. Denote elements of $\mathfrak{v}$ by (p, q) with $p, q \in A$. For $q \in A$, let q_R and q_I denote the real and imaginary parts of q. Using formulas (1.3) and (1.7), we see that the bracket operation in $\mathfrak{g}_{1,1}$ is given by

$$[(p, q), (p', q')] = (pp' + q'q)_I - 2p_I p'_R - 2q_I q'_R.$$

The subspaces $\mathfrak{v}_1$ and $\mathfrak{v}_2$ given by

$$\mathfrak{v}_1 = \{(q, \overline{q}): q \in A\} \quad \text{and} \quad \mathfrak{v}_2 = \{(q, -\overline{q}): q \in A\}$$

are abelian subalgebras of $\mathfrak{g}_{1,1}$ contained in $\mathfrak{v}$ and hence are flat.

Next the bracket operation in $\mathfrak{g}_{2,0}$ is given by

$$[(p, q), (p', q')] = -(p\overline{p}' + q\overline{q}')_I.$$

We show that $\mathfrak{v}$ contains no four-dimensional abelian subalgebras of $\mathfrak{g}_{2,0}$. Suppose that $\mathfrak{m}$ is such an abelian subalgebra. Denote by $\mathfrak{a}$ and $\mathfrak{b}$ the subspaces $A \times \{0\}$ and $\{0\} \times A$ of $\mathfrak{v}$, respectively, and let $\pi_{\mathfrak{a}}$ and $\pi_{\mathfrak{b}}$ denote the projections of $\mathfrak{v}$ onto $\mathfrak{a}$ and $\mathfrak{b}$. Since $\mathfrak{a}$ and $\mathfrak{b}$ commute and neither subspace contains any abelian subalgebras of $\mathfrak{g}_{2,0}$ of dimension greater than one, both $\pi_{\mathfrak{a}}$ and $\pi_{\mathfrak{b}}$ restrict to isomorphisms on $\mathfrak{m}$. Thus we obtain a vector space isomorphism

$$\Phi = \pi_{\mathfrak{b}} \circ \pi_{\mathfrak{a}}^{-1} : \mathfrak{a} \to \mathfrak{b}$$

satisfying $[\Phi(X), \Phi(Y)] = -[X, Y]$. We may view Φ as a vector space isomorphism of the quaternions A satisfying

$$\Phi(p)\overline{\Phi(p')} = -p\overline{p}' \quad \text{for all } p, p' \in A.$$

An elementary argument shows that no such isomorphism exists. The proposition follows.

Corollary 3.5. *The* 11*-dimensional manifold* $G_{1,1}$ *admits three mutually orthogonal foliations corresponding to the distributions* $\mathfrak{v}_1$, $\mathfrak{v}_2$, *and* $\mathfrak{z}$. *The leaves of these foliations are flat submanifolds of* $G_{1,1}$ *of dimension* 4, 4, *and* 3, *respectively. The first two foliations are tangent to the negative eigenspace of the Ricci tensor; the third is tangent to the positive eigenspace.*

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